Svar på opgave 333 (Oktober 2016)

Opgave:

ΔABC er retvinklet med $C = 90°$.

Vis, at $\dfrac{1}{2c+a+b}+\dfrac{1}{c+2a+b}+\dfrac{1}{c+a+2b}<\dfrac{c}{2ab}$.

Besvarelse:

1. metode

Vi har, at $c^2 = a^2 + b^2 \geq 2ab$ og får

$$c(2c + a + b) = 2c^2 + c(a + b) > 2c^2 + c \cdot c = 3c^2 \geq 6ab ,$$

$$c(c + 2a + b) = c^2 + c(2a + b) \geq 2ab + \sqrt{2ab} \cdot 2\sqrt{2ab} = 6ab .$$

Her har vi benyttet, at $p + q \geq 2\sqrt{pq}$ med $p = 2a$ og $q = b$. Endelig er

$$c(c + a + 2b) = c^2 + c(a + 2b) \geq 2ab + \sqrt{2ab} \cdot 2\sqrt{2ab} = 6ab .$$

Altså er

$$\frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b}=\frac{c}{c(2c+a+b)}+\frac{c}{c(c+2a+b)}+\frac{c}{c(c+a+2b)}$$

$$\leq \frac{c}{6ab}+\frac{c}{6ab}+\frac{c}{6ab}=\frac{c}{2ab} .$$

2. metode

Det gælder, at

$$c \geq \sqrt{2ab} \quad , \quad a+b \geq 2\sqrt{ab} \quad , \quad 2a+b \geq 2\sqrt{2ab} \quad , \quad a+2b \geq 2\sqrt{2ab} \quad ,$$

så vi får

$$\frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b} \leq \frac{1}{2\sqrt{2ab}+2\sqrt{ab}}+\frac{2}{\sqrt{2ab}+2\sqrt{2ab}}$$

$$\Leftrightarrow \quad \frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b} \leq \frac{1}{\sqrt{2ab}} \cdot \frac{1}{2+\sqrt{2}}+\frac{1}{\sqrt{2ab}} \cdot \frac{2}{1+2}$$

$$\Leftrightarrow \quad \frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b} \leq \frac{1}{\sqrt{2ab}} \cdot \left(\frac{1}{2+\sqrt{2}}+\frac{2}{3}\right) . \qquad (1)$$

Ved hjælp af lidt algebra får vi, at

$$\frac{1}{2+\sqrt{2}}+\frac{2}{3}=\frac{10-3\sqrt{2}}{6}<1 \qquad \text{fordi} \qquad 10-3\sqrt{2}<6 \quad \Leftrightarrow \quad 4<3\sqrt{2}=\sqrt{18} .$$

Altså er

$$\frac{1}{\sqrt{2ab}}\cdot\left(\frac{1}{2+\sqrt{2}}+\frac{2}{3}\right)<\frac{1}{\sqrt{2ab}}=\frac{\sqrt{2ab}}{2ab}\le\frac{c}{2ab}\ ,$$

så vi af (1) får det ønskede.

3. metode

Løsningen mellem aritmetisk og geometrisk middeltal giver

$$2c+a+b>3\sqrt[3]{2abc}\ ,\ c+2a+b>3\sqrt[3]{2abc}\quad ,\quad c+a+2b>3\sqrt[3]{2abc}\ .$$

Dermed er

$$\frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b}\le\frac{3}{3\sqrt[3]{2abc}}=\frac{1}{\sqrt[3]{2abc}}\ . \qquad (2)$$

Nu er

$$c^4=(a^2+b^2)^2\ \ge\ 4a^2b^2\ ,$$

hvoraf

$$4a^2b^2\ \le\ c^4\quad\Leftrightarrow\quad 8a^3b^3\ \le\ 2abc^4\quad\Leftrightarrow\quad\frac{1}{2abc}\le\frac{c^3}{8a^3b^3}\ ,$$

så at

$$\frac{1}{\sqrt[3]{2abc}}\le\frac{c}{2ab}\ ,$$

og det ønskede fås af (2).

4. metode

Uligheden mellem aritmetisk og harmonisk middeltal giver

$$\frac{4}{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}}\le\frac{p+q+r+s}{4}\ ,$$

hvoraf

$$\frac{1}{2c+a+b}=\frac{1}{c+c+a+b}\le\frac{\frac{1}{c}+\frac{1}{c}+\frac{1}{a}+\frac{1}{b}}{16}\ ,$$

$$\frac{1}{c+2a+b}=\frac{1}{c+a+a+b}\le\frac{\frac{1}{c}+\frac{1}{a}+\frac{1}{a}+\frac{1}{b}}{16}\ ,$$

$$\frac{1}{c+a+2b}=\frac{1}{c+a+b+b}\le\frac{\frac{1}{c}+\frac{1}{a}+\frac{1}{b}+\frac{1}{b}}{16}.$$

Dermed er

$$\frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b}\le\frac{1}{16}\left(\frac{4}{a}+\frac{4}{b}+\frac{4}{c}\right)=\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Vi omskriver sådan

$$\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{4abc}\big(ab+c(a+b)\big)\le\frac{1}{4abc}\left(\frac{a^2+b^2}{2}+c\sqrt{2(a^2+b^2)}\right)$$

$$=\frac{1}{4abc}\left(\frac{c^2}{2}+c^2\sqrt{2}\right)=\frac{c}{4ab}\left(\frac{1}{2}+\sqrt{2}\right)<\frac{2c}{4ab}=\frac{c}{2ab}\ .$$

5. metode **(Asger Olesen, Tønder)**

Vi benytter, at $(x + y)^2 \geq 4xy$, fordi denne ulighed er ensbetydende med $(x - y)^2 \geq 0$. Vi får, at

$$(2c + a + b)^2 = (a + c + b + c)^2 \geq 4(a + c)(b + c)$$

$$\Leftrightarrow \quad \frac{1}{4}\cdot\frac{2c+a+b}{(a+c)(b+c)} \geq \frac{1}{2c+a+b} \quad \Leftrightarrow \quad \frac{1}{4}\cdot\left(\frac{1}{a+c}+\frac{1}{b+c}\right) \geq \frac{1}{2c+a+b}\ .$$

Tilsvarende er

$$\frac{1}{4}\cdot\left(\frac{1}{b+a}+\frac{1}{c+a}\right) \geq \frac{1}{c+2a+b} \quad \text{og} \quad \frac{1}{4}\cdot\left(\frac{1}{a+b}+\frac{1}{c+b}\right) \geq \frac{1}{c+a+2b}\ .$$

Addition giver

$$\frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b} \leq \frac{1}{4}\cdot\left(\frac{1}{a+c}+\frac{1}{b+c}+\frac{1}{b+a}+\frac{1}{c+a}+\frac{1}{a+b}+\frac{1}{c+b}\right)$$

$$\frac{1}{2c+a+b}+\frac{1}{c+2a+b}+\frac{1}{c+a+2b} \leq \frac{1}{2}\cdot\left(\frac{1}{a+c}+\frac{1}{b+c}+\frac{1}{c+a}\right).$$

Det er derfor nok at vise, at

$$\frac{1}{a+c}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{c}{ab}.$$

Denne ulighed er ensbetydende med

$$\frac{(b+c)(c+a)+(a+b)(c+a)+(a+b)(b+c)}{(a+b)(b+c)(c+a)} < \frac{c}{ab}$$

$$\Leftrightarrow \quad \frac{a^2+b^2+c^2+3(ab+bc+ca)}{(a+b)(b+c)(c+a)} < \frac{c}{2ab}\ .$$

Ved hjælp af uligheden mellem aritmetisk og geometrisk middeltal fås

$$(a+b)(b+c)(c+a) \leq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ac} = 8abc\ ,$$

hvoraf

$$\frac{1}{(a+b)(b+c)(c+a)} \leq \frac{1}{8abc}$$

$$\Leftrightarrow \quad \frac{a^2+b^2+c^2+3(ab+bc+ca)}{(a+b)(b+c)(c+a)} \leq \frac{a^2+b^2+c^2+3(ab+bc+ca)}{8abc}\ .$$

Vi kan derfor nøjes med at vise, at

$$\frac{a^2+b^2+c^2+3(ab+bc+ca)}{8abc} < \frac{c}{ab}\ .$$

Denne ulighed omskrives ved ensbetydende regninger og vi benytter undervejs, at trekanten er retvinklet:

$$\frac{a^2+b^2+c^2+3(ab+bc+ca)}{8abc} < \frac{c}{ab} \quad \Leftrightarrow \quad a^2+b^2+c^2+3(ab+bc+ca) < 8c^2$$

$$\Leftrightarrow \quad 3(ab + bc + ca) < 6c^2 \quad \Leftrightarrow \quad 2(ab + bc + ca) < 4c^2$$

$$\Leftrightarrow \quad (a+b+c)^2 - a^2 - b^2 - c^2 < 4c^2 \quad \Leftrightarrow \quad (a+b+c)^2 < 6c^2$$

$$\Leftrightarrow \quad a+b+c < c\sqrt{6} \quad \Leftrightarrow \quad \tfrac{a}{c}+\tfrac{b}{c}+1<\sqrt{6} \quad \Leftrightarrow \quad \sin A + \cos A < \sqrt{6}-1$$

$$\Leftrightarrow \quad \sqrt{2}\cdot\sin(45^\circ + A) < \sqrt{6}-1 \quad \Leftrightarrow \quad \sin(45^\circ + A) < \frac{\sqrt{6}-1}{\sqrt{2}} \approx 1{,}0249 \ ,$$

hvilket er sandt.